ONE-DIMENSIONAL MOTION

Ch. 2 Motion along a line

Kinematics

- Mechanics, the study of motion, splits into two parts:
  - **Kinematics:** the **description** of motion
    - *Given acceleration*, find equations for position and velocity as functions of time
  - **Dynamics:** the **explanation** of motion
    - What *causes* the acceleration? (Answer: forces)
- This chapter and the next: kinematics
- Chapter 4: the beginning of dynamics

Notation

- $t =$ time (in seconds)
- $x =$ position (usually in meters) at time t
- $t_0 =$ initial time (will usually take $t_0 = 0$)
- $x_0 =$ initial position at time t_0

- Δ (capital Greek delta) = "Change in"

  = (final value) – (initial value)

- For example:
  - $\Delta x = x - x_0,$ and $\Delta t = t - t_0$

Speed and velocity

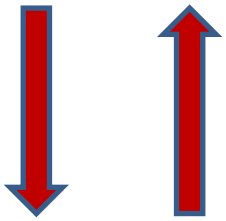
- **Velocity v** is the **rate of change** of position, i.e. the change in position per unit time
- Units of v are meters per second (m/s)
- Velocity can be positive or negative. The sign indicates direction:
  - Positive v means motion upward or to the right
  - Negative v means motion downward or to the left

Horizontal motion:

$v < 0$ $v > 0$

Vertical motion:

$v > 0$

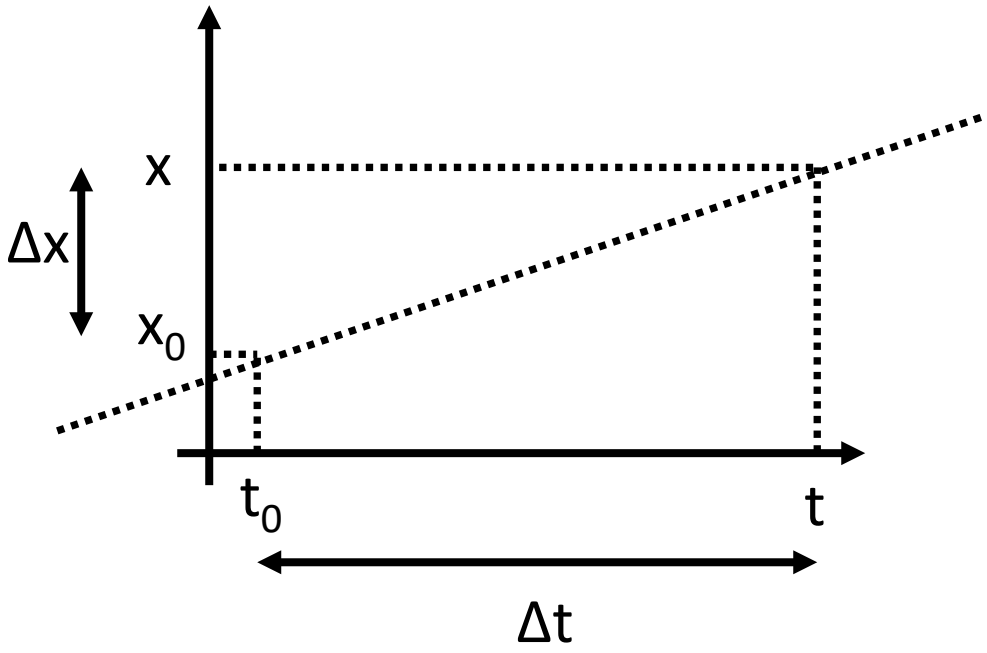


$v > 0$

Speed and velocity

- The sign of velocity gives the **direction** object is moving. The absolute value gives **how fast** it is moving.
- This absolute value is the **speed**: $v = |\boldsymbol{v}|$ (speed: v; velocity: $\boldsymbol{v}$)
- Note that the velocity is bold print, the speed is not. (More on the meaning of this in the next chapter.)
- To reiterate:
  - Velocity can be negative (sign indicates direction)
  - Speed is always positive (or zero)

Average Velocity



- Consider a time interval, from t_0 to t.
- What is the average velocity over this interval?

Average velocity: $$\overline{v} = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0}$$ (bar = average)

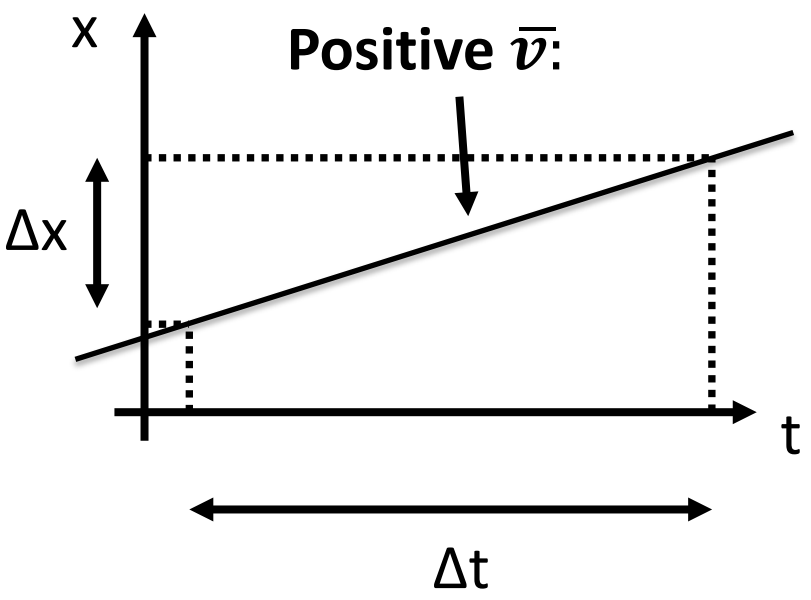
Average speed: $$|\overline{v}| = \left|\frac{\Delta x}{\Delta t}\right| = \left|\frac{x - x_0}{t - t_0}\right|$$

Average Velocity

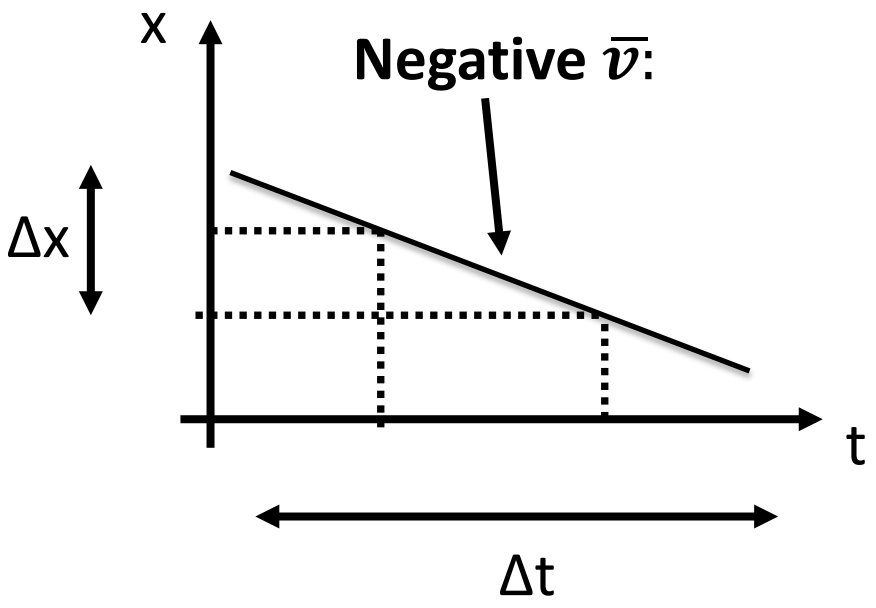
Average velocity:

$$\overline{\boldsymbol{v}} = \frac{\Delta x}{\Delta t} = \frac{x - x_0}{t - t_0}$$

- This is the **slope** of the straight line joining the initial and final times



Object is moving to right (on average)

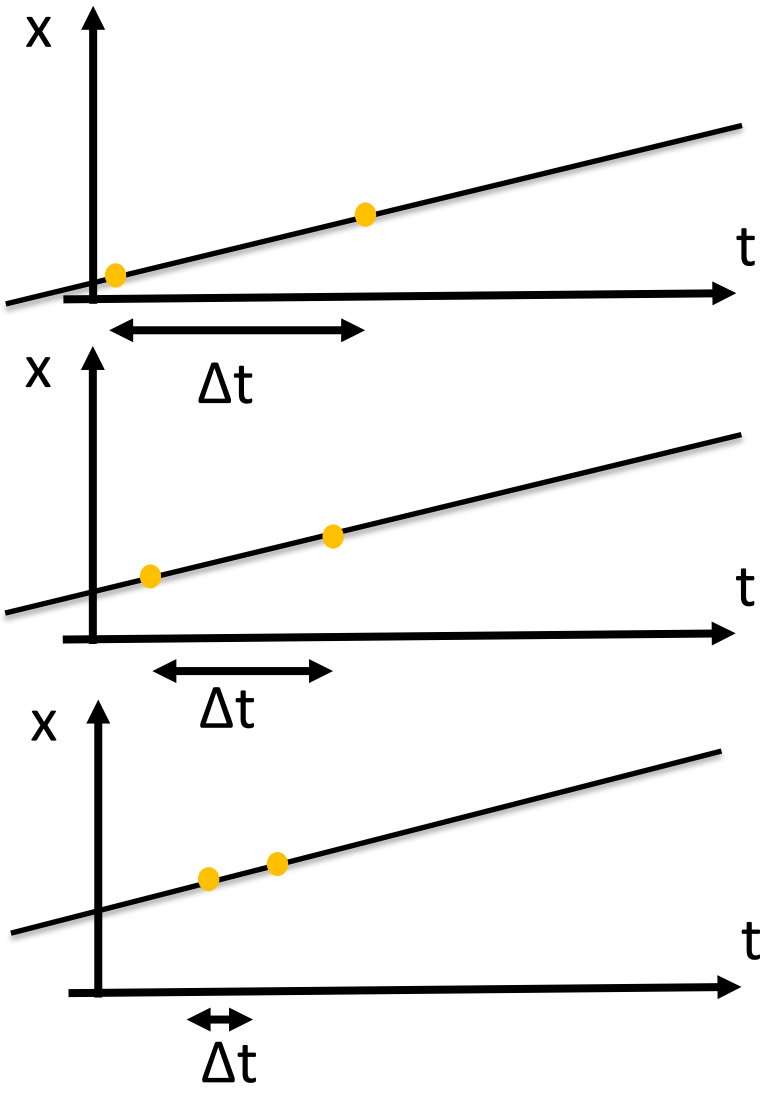


Object is moving to left (on average)

Instantaneous Velocity

- **Instantaneous velocity**, $\boldsymbol{v}$, is the velocity at a particular moment, not average over an interval of time. It is a function of time: $\boldsymbol{v(t)}$
- Slide the endpoints of the interval closer together so that Δx and Δt become infinitesimal:

$$\Delta x \rightarrow dx\,, \qquad \Delta t \rightarrow dt$$

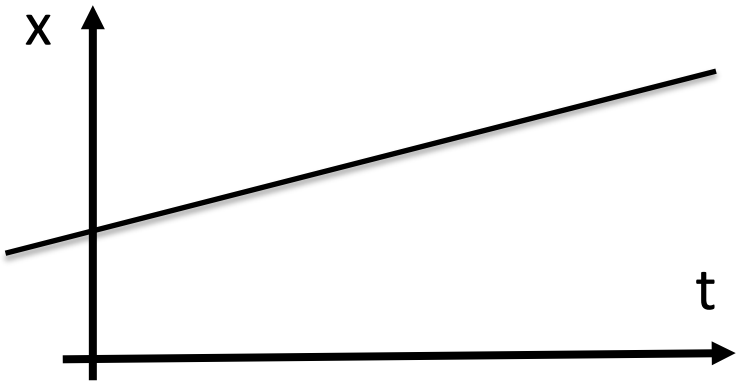


Instantaneous Velocity

- As you take the infinitesimal limit, the ratio Δx/Δt becomes a **derivative.**
- **Instantaneous velocity:**

$$\boldsymbol{v} = \frac{dx}{dt} = \lim_{\Delta t \to 0} \frac{\Delta x}{\Delta t}$$

- The instantaneous velocity measures the slope of the line **tangent** to the curve at time t



Average & Instantaneous Velocities

- Note that:
  - $\overline{v}$ is the average over some finite **interval**, Δt.
  - v is value at a **precise moment**, t.
- When people say "velocity" without any qualifier, they usually mean the instantaneous velocity.
- If position x(t) is known as function of time, v(t) is found by taking derivative.
- We want to minimize use of calculus for now, so we will focus mostly on cases where we can find velocity without taking derivatives.

Example 1

- A jogger travels 400 m in 2 minutes. What is his average speed in meters per second?

$$|\bar{v}| = \left|\frac{\Delta x}{\Delta t}\right| = \left|\frac{400\ m}{2\ min}\right| \cdot \left(\frac{1\ min}{60\ s}\right) = \boxed{3.33\ \frac{m}{s}}$$

Conversion factor

- Note that we are given no information about direction, so we can only find **speed**, not **velocity**.

Example 2

- An airplane travels at 500 mph (average speed). How far (in miles) does it travel in 20 minutes?
- Given quantities: $|\bar{v}| = 500\ \frac{miles}{hour}$

$$\Delta t = 20\ min.\left(\frac{1\ hr}{60\ min.}\right) = .333\ hr$$

Conversion factor

- Rearrange $|\bar{v}| = \left|\frac{\Delta x}{\Delta t}\right|$ to solve for distance:

$$|\Delta x| = |\bar{v}\,\Delta t| = \left(500\ \frac{miles}{hr}\right)(.333\ hr) = \boxed{167\ miles}$$

Acceleration

- One more basic kinematic quantity is needed: **Acceleration** is the rate of change of the velocity:
  - Average acceleration: $\bar{\boldsymbol{a}} = \dfrac{\Delta \boldsymbol{v}}{\Delta t} = \dfrac{\boldsymbol{v} - \boldsymbol{v}_0}{t - t_0}$
  - Instantaneous acceleration: $\boldsymbol{a}(\boldsymbol{t}) = \dfrac{d\boldsymbol{v}}{dt} = \lim\limits_{\Delta t \to 0} \dfrac{\Delta \boldsymbol{v}}{\Delta t}$
- Just as velocity was the slope of x vs. t graph, acceleration is the slope of the v vs. t graph.
- Units of acceleration: $\dfrac{m}{s^2}$ (or $\dfrac{ft}{s^2}$, $\dfrac{mile}{hr^2}$, etc.)

Acceleration

- The sign of acceleration tells the direction of change of velocity:
  - $\boldsymbol{a} > 0 \rightarrow$ velocity is increasing
  - $\boldsymbol{a} < 0 \rightarrow$ velocity is decreasing
- Note that the previous lines say **velocity**, not **speed**.
- If velocity is negative, then:
  - $\boldsymbol{a} > 0 \rightarrow \boldsymbol{v}$ is increasing (becoming less negative), so speed is decreasing (slowing down)
  - $\boldsymbol{a} < 0 \rightarrow \boldsymbol{v}$ is decreasing (becoming more negative), so speed is decreasing (speeding up)

Position, velocity, and acceleration

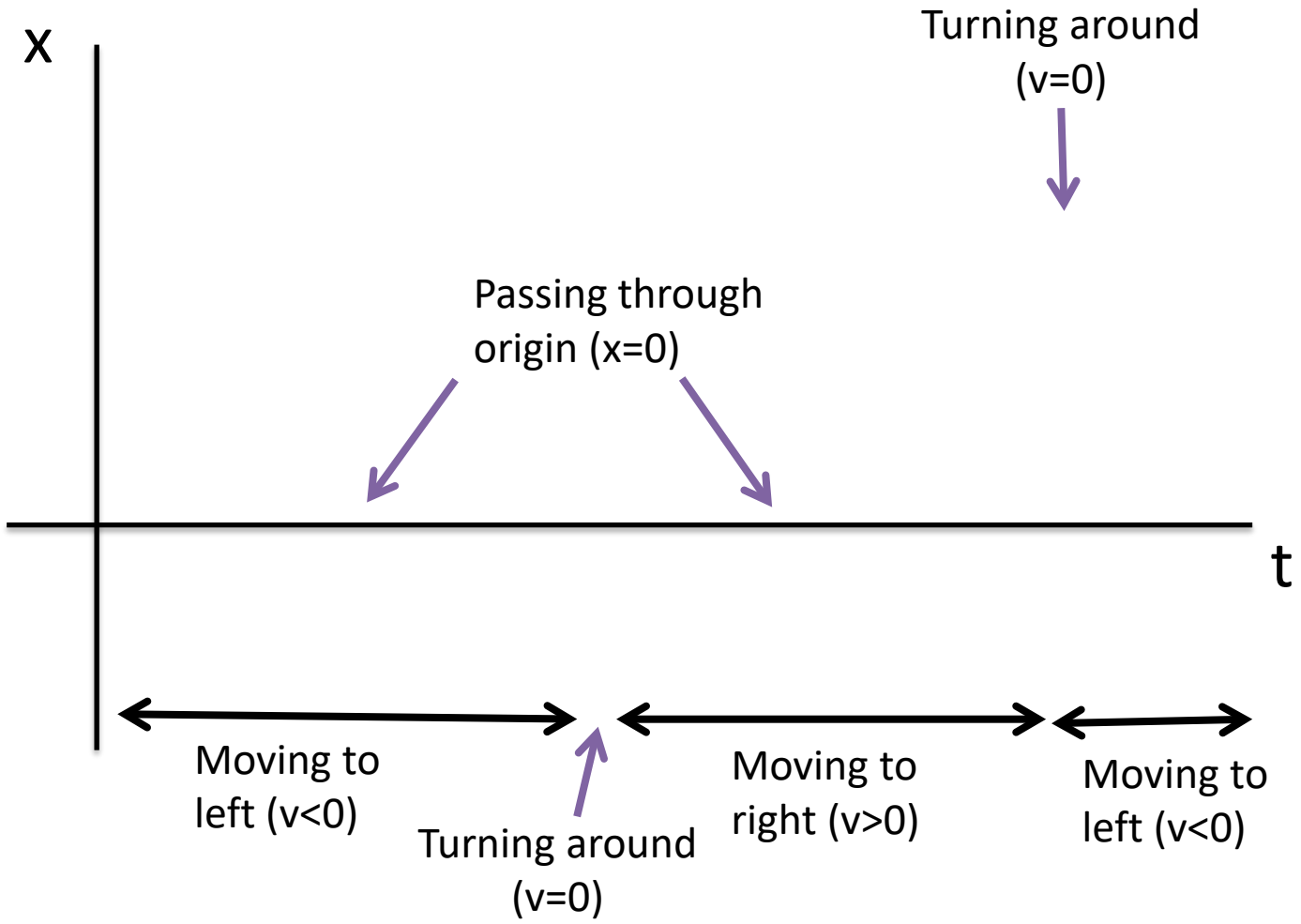
- We now have all of our basic kinematic quantities: position, velocity, and acceleration.
- In general, all three are time-dependent.
- Summarizing their relations:

$\boldsymbol{v(t)} = \frac{dx}{dt}$ = rate of change of x = slope of x vs. t graph

$\boldsymbol{a(t)} = \frac{d\boldsymbol{v}}{dt} = \frac{d^2x}{dt^2}$ = rate of change of v = slope of v vs. t graph

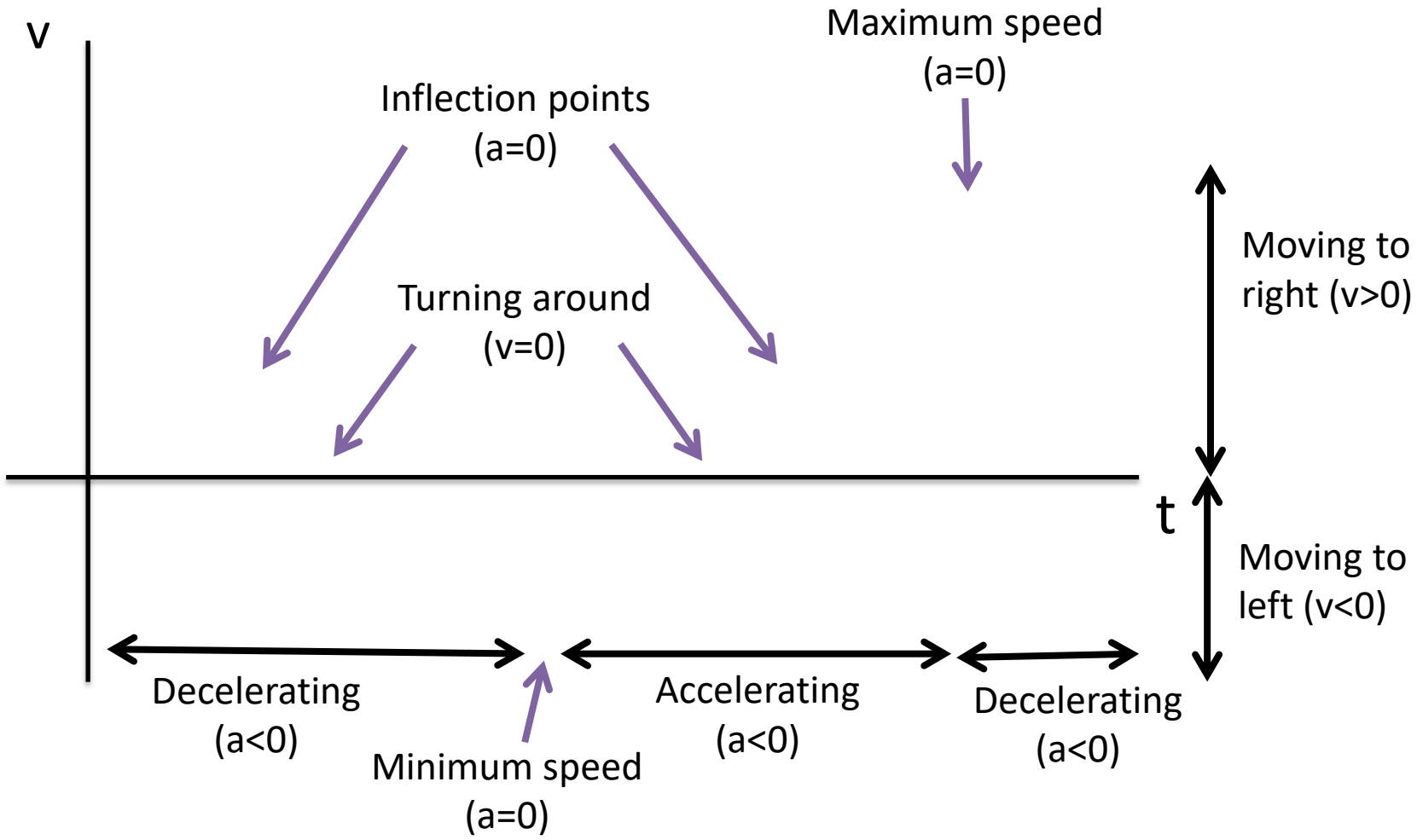
= rate at which slope of x vs. t graph is changing (curvature)

x versus t graph



- **Important point:** $v = 0$ at turning points (maxima and minima)

v versus t graph



- a=0 at inflection points, where graph switches between concave up and concave down

Example 1

- A car accelerates from 0 to 60 mph in 7 seconds. What is its average acceleration in ft/s^2 ?
- Known: $\boldsymbol{v}_0 = 0, \qquad \Delta t = 7\ s$

$$\boldsymbol{v} = \left(60\frac{mile}{hr}\right)\left(\frac{5280\ ft}{1\ mile}\right)\left(\frac{1\ hr}{3600\ s}\right) = 88\ \frac{ft}{s}$$

- So: $\bar{\boldsymbol{a}} = \dfrac{\Delta \boldsymbol{v}}{\Delta t} = \dfrac{\boldsymbol{v} - \boldsymbol{v}_0}{\Delta t} = \dfrac{(88-0)\,{}^{ft}/_{s}}{7\ s} = \boxed{12.57\ \dfrac{ft}{s^2}}$

The fine print:

- We assumed that the motion is to right, so we could take $v > 0$ and not worry about minus signs.
- If we are told that a is constant, then the bar can be dropped: if something is constant, then its average and instantaneous values are the same.

Example 2

- Same car as last example now decelerates from $88\ \frac{ft}{s}$ to rest in 4 seconds. What is its average acceleration in ft/s^2 ?
- Known: $\boldsymbol{v}_0 = 88\ \frac{ft}{s}, \quad \Delta t = 4\ s, \quad \boldsymbol{v} = 0$
- So:

$$\overline{\boldsymbol{a}} = \frac{\Delta \boldsymbol{v}}{\Delta t} = \frac{\boldsymbol{v} - \boldsymbol{v}_0}{\Delta t} = \frac{(0 - 88)\ ^{ft}/_{s}}{4\ s} = \boxed{-22.0\ \frac{ft}{s^2}}$$

- The car is now slowing down (velocity and speed are decreasing)

Special Case: Constant Acceleration

- Now we assume **a = constant** (not a function of time). we can write explicit formulas for position and velocity as functions of time.
- **Constant acceleration formulas:**

$$1.\ x = x_0 + v_0 t + \frac{1}{2} a t^2$$

$$2.\ v = v_0 + at$$

$$3.\ v^2 = v_0^2 + 2a(x - x_0)$$

x and v are position and velocity at time t

x_0 and v_0 are position and velocity at time 0

Constant acceleration equations

1. $x = x_0 + v_0 t + \frac{1}{2} a t^2$
2. $v = v_0 + at$
3. $v^2 = v_0^2 + 2a(x - x_0)$

- Note:
  - Differentiating 1st equation gives 2nd one. Differentiating again gives a.
  - Integrating a once gives eq. 2 and integrating again gives eq. 1. x_0 and v_0 are integration constants
  - 3rd equation follows from plugging eq. 2 into eq. 1 to eliminate t.
  - These hold **only** if a = constant.
  - Note that *distance = (speed) x (time)* is true **only** if a = 0.

Example 1

- A skier descends slope. with initial speed $v_0 = 8\ \frac{m}{s}$. She decelerates *at a constant rate*, coming to a stop after a distance of $\Delta x = x - x_0 = 400\ m$.

(a) What is her acceleration?

$v^2 = v_0^2 + 2a(x - x_0)$ $\rightarrow$ Solve for a.

$$a = \frac{v^2 - v_0^2}{2(x - x_0)} = \frac{0^2 - (8\frac{m}{s})^2}{2(400\ m)} = \boxed{-.08\ \frac{m}{s^2}}$$

(b) How long does it take her to stop?

$$v = v_0 + at \quad \rightarrow \quad t = \frac{v - v_0}{a} = \frac{0 - (8\ m/s)}{.8\ m/s^2} = \boxed{100\ s}$$

Example 2

- Initially, car has $v_0 = -15\ \frac{m}{s}$, but it accelerates at $a = +2\ \frac{m}{s^2}$.

(a) At what time does car turn around?

$$v = v_0 + at \quad \rightarrow \quad t = \frac{v - v_0}{a} = \frac{0 - (-15\ \frac{m}{s})}{2\ \frac{m}{s^2}} = \boxed{7.5\ s}$$

v = 0 at turning point

(b) Where is it at this time? (Assume $x_0 = 0.$)

$$x = x_0 + v_0 t + \frac{1}{2}at^2 = 0 + (-15)(7.5) + \frac{1}{2}(2)(7.5)^2 = \boxed{-56.25\ m}$$

(c) At what time will it return to the origin?

$$\cancel{x} = \cancel{x_0} + v_0 t + \tfrac{1}{2}at^2 \quad \rightarrow \quad 0 = \left(v_0 + \tfrac{1}{2}at\right)t$$

Solutions: (i) t=0 (We already knew this)

(ii) $t = -\frac{2v_0}{a} = -\frac{2(-15)}{2} = \boxed{15s}$

Example 3

- A Porsche brakes from $v_0 = 75\frac{km}{hr}$ $\left(= 20.83\ \frac{m}{s}\right)$ to $v = 45\frac{km}{hr}$ $\left(= 12.5\ \frac{m}{s}\right)$, *with constant acceleration*. During this time, the car travels distance $x - x_0 = 88\ m$.

(a) What is the acceleration?

$v^2 = v_0^2 + 2a(x - x_0)$ → Solve for a.

$$a = \frac{v^2 - v_0^2}{2(x - x_0)} = \frac{(12.5\frac{m}{s})^2 - (20.83\frac{m}{s})^2}{2(88\ m)} = \boxed{-1.58\ \frac{m}{s^2}}$$

(b) How long did it take to reach final speed?

$$v = v_0 + at \quad \rightarrow \quad t = \frac{v - v_0}{a} = \frac{(12.5\frac{m}{s}) - (20.83\frac{m}{s})}{-1.58\frac{m}{s^2}} = \boxed{5.27\ s}$$

Example continued

Example 3 (continued)

(c) If the acceleration continues, how long does it take to bring car to rest?

We know: $v = 0, \quad v_0 = 75\ \frac{km}{hr} = 20.83\ \frac{m}{s}, \qquad a = -1.58\ \frac{m}{s^2}$

So:

$$\mathrm{v} = v_0 + \mathrm{at} \quad \rightarrow \quad t = \frac{v - v_0}{a} = \frac{0 - (20.83\ m/s)}{(-1.58\frac{m}{s^2})} = \boxed{13.18\ s}$$

(d) How far does the car travel before coming to rest?

$$\Delta x = x - x_0 = v_0 t + \frac{1}{2} a t^2$$

$$= \left(20.83\ \frac{m}{s}\right)(13.18\ s) + 2\left(-1.58\frac{m}{s^2}\right)(13.18\ s)^2$$

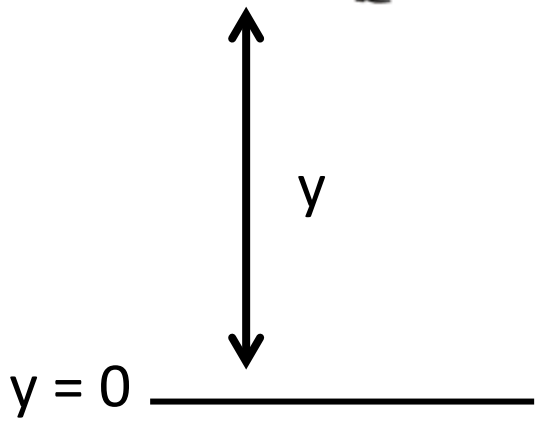
$$= \boxed{137.31\ m} \quad \text{(about 450 ft)}$$

Freely-falling objects

- The most important example of constant acceleration is an object in free fall.
- An object near the surface of Earth is falling freely under influence of gravity.
- **No forces** except gravity are acting.
- Then the acceleration will be constant.

(see next slide)

- For vertical motion, we will use y to denote height (instead of x).
- Take origin at ground level (so y=0 at ground).



Freely-falling objects

- The acceleration due to gravity is **constant**.
- Because it is **downward**, it should be negative.

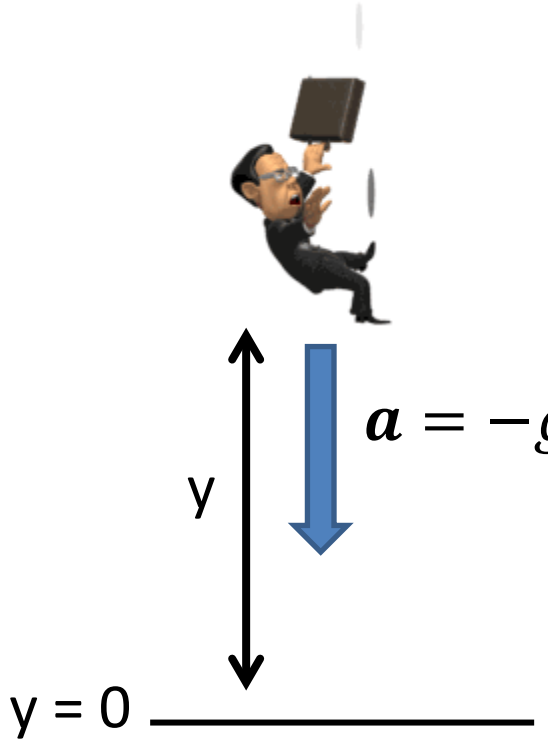
Gravitational acceleration:

$$\boldsymbol{a} = -g,$$

where:

$$g = 9.8\ \frac{m}{s^2} = 32\ \frac{ft}{s^2}$$

- **Warning:** $\boldsymbol{a}$ is constant and equal to this value **only** near the Earth's surface.
- We'll look at more general cases in a later chapter



Free fall

- Because gravitational acceleration is constant, we can use the constant acceleration equations.
- We just make the replacements:

$$a \rightarrow -g, \qquad x \rightarrow y$$

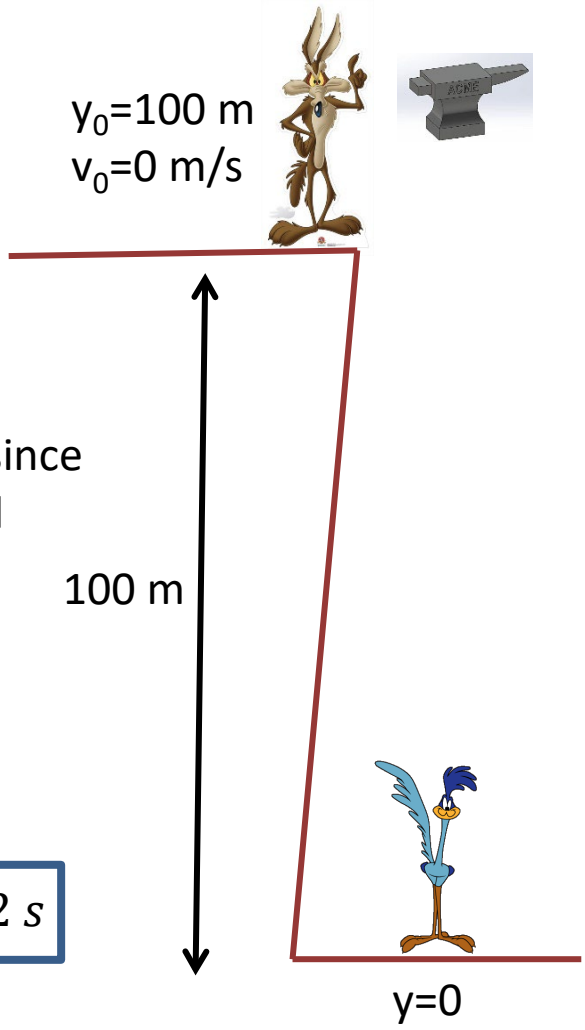
- Result:

1. $y = y_0 + v_0 t - \frac{1}{2} g t^2$
2. $v = v_0 - gt$
3. $v^2 = v_0^2 - 2g(y - y_0)$

- Note: only **differences** in height matter. Can choose origin (y=0) anywhere you want, and measure differences from there.

Example 1

- The coyote is at the top of a cliff, 100 m high. He drops an Acme™ anvil, which starts from rest.



(a) What is the speed of the anvil at the bottom of the cliff?

$$v = -\sqrt{v_0^2 - 2g(y - y_0)}$$

← **Negative** root used, since v must be **downward**

$$= -\sqrt{0 - 2\left(9.8\,\frac{m}{s^2}\right)(0 - 100m)} \quad = \boxed{-44.3\,\frac{m}{s}}$$

(b) How long does it take to fall?

$$v = v_0 - gt \quad \rightarrow \quad t = \frac{v - v_0}{g} = \frac{0 - \left(-44.3\,\frac{m}{s}\right)}{9.8\,\frac{m}{s^2}} = \boxed{4.52\ s}$$

Note that we did **not** need to know the mass of the anvil. Objects of different mass fall at the same rate (ignoring air resistance).

Example 2

- Throw a ball straight upward with initial speed of 15 m/s. (Can take initial height to be $y_0 = 0.$)

(a) Find the height and velocity one second later.

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 = 0 + \left(15\ \frac{m}{s}\right)(1s) - \frac{1}{2}\left(9.8\ \frac{m}{s^2}\right)(1s)^2 = \boxed{10.1m}$$

$$v = v_0 - gt = \left(15\frac{m}{s}\right) - \left(9.8\ \frac{m}{s^2}\right)(1s) = \boxed{5.2\frac{m}{s}}$$

(b) Find the height and velocity after 2 seconds.

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 = 0 + \left(15\ \frac{m}{s}\right)(2s) - \frac{1}{2}\left(9.8\ \frac{m}{s^2}\right)(2s)^2 = \boxed{10.4m}$$

$$v = v_0 - gt = \left(15\frac{m}{s}\right) - \left(9.8\ \frac{m}{s^2}\right)(2s) = \boxed{-4.6\frac{m}{s}}$$

Example continued

Example 2 (continued)

(c) Find the maximum height. At what time is it reached?

To find maximum height, set $v = 0$ at that point.

$v = v_0 - gt \quad \rightarrow \quad 0 = v_0 - gt \quad \rightarrow$ Solve for t.

$$t = \frac{v_0}{g} = \frac{15\ m/s}{9.8\ m/s^2} = \boxed{1.53\ s}$$

$$\mathrm{y} = y_0 + v_0 t - \frac{1}{2} g t^2 = 0 + \left(15\ \frac{m}{s}\right)(1.53s) - \frac{1}{2}\left(9.8\ \frac{m}{s^2}\right)(1.53s)^2$$

$$= \boxed{11.48\ m}$$

Consistency check: From parts a and b, we know the time of maximum height has to between 1s and 2s.

Example continued

Example 2 (continued)

(d) How long before the ball returns to initial height?

Now: $y = y_0 = 0$

$$y = y_0 + v_0 t - \frac{1}{2} g t^2 \quad \rightarrow \quad 0 = 0 + v_0 t - \frac{1}{2} g t^2 = \left(v_0 - \frac{1}{2} g t \right) t$$

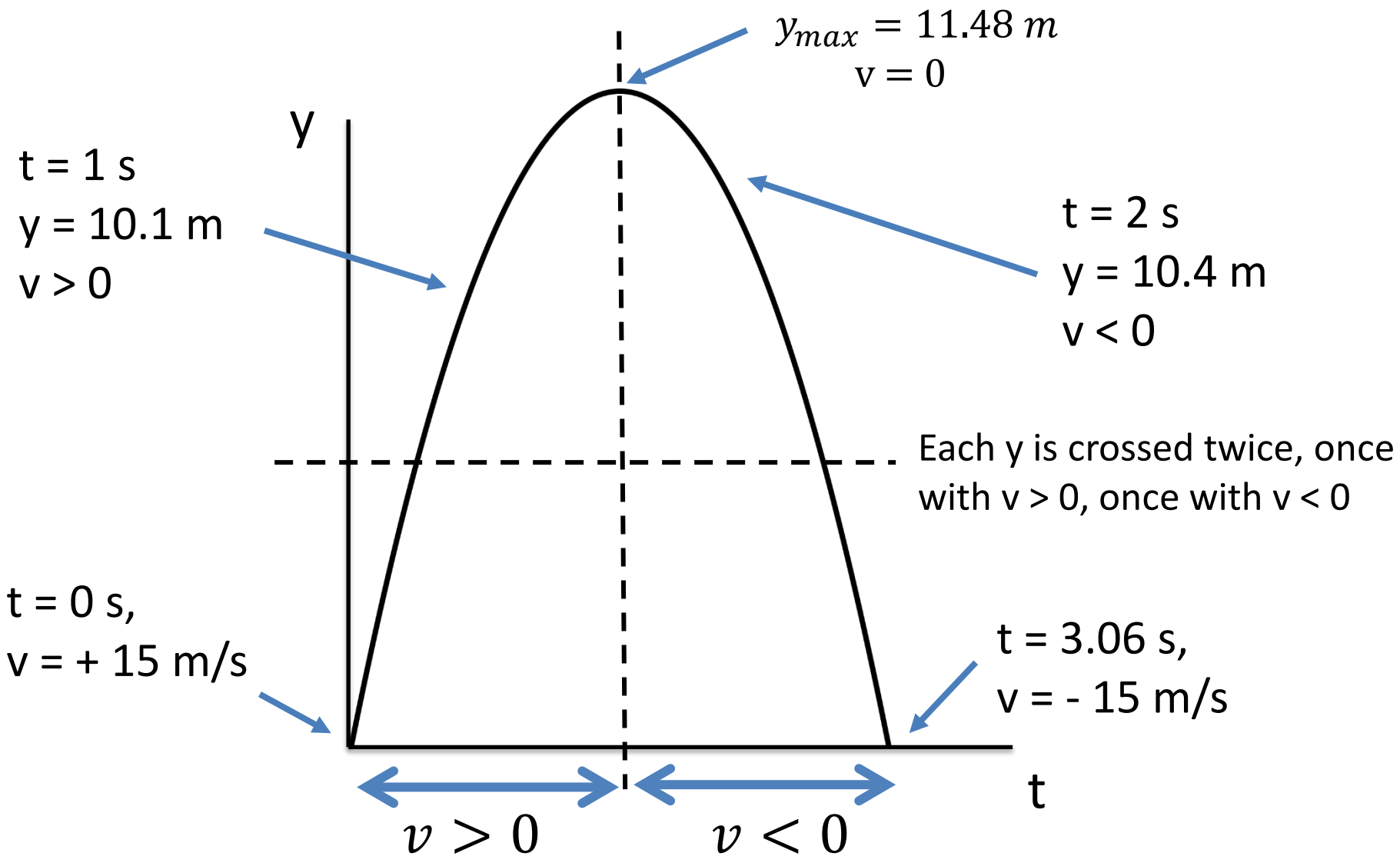
$$= 0 + \left(15 \frac{m}{s} \right) (1.53 s) - \frac{1}{2} \left(9.8 \frac{m}{s^2} \right) (1.53 s)^2$$

This has two solutions. One is t=0. The other:

$$t = \frac{2 v_0}{g} = \frac{2(15 \frac{m}{s})}{9.8 \frac{m}{s^2}} = \boxed{3.06\ s}$$

Example continued

Example 2 (continued)



Example 3

- Krazy the Kaffienated Kat wants to catch a bee 2m above the ground. What must her initial launch speed be?

Know: $v = 0$ at max. height, $y = 2m$. Initially: $y_0 = 0$, $v_0 = ?$

We neither have nor want information about time, so use the equation that doesn't involve t.

$$v^2 = v_0^2 - 2g(y - y_0) \qquad \rightarrow \qquad 0 = v_0^2 - 2g(y - y_0)$$

Solve for v_0:

$$v_0 = +\sqrt{2g(y - y_0)} = +\sqrt{2(9.8\ \frac{m}{s^2})(2\ m)} = \boxed{6.26\ \frac{m}{s}}$$

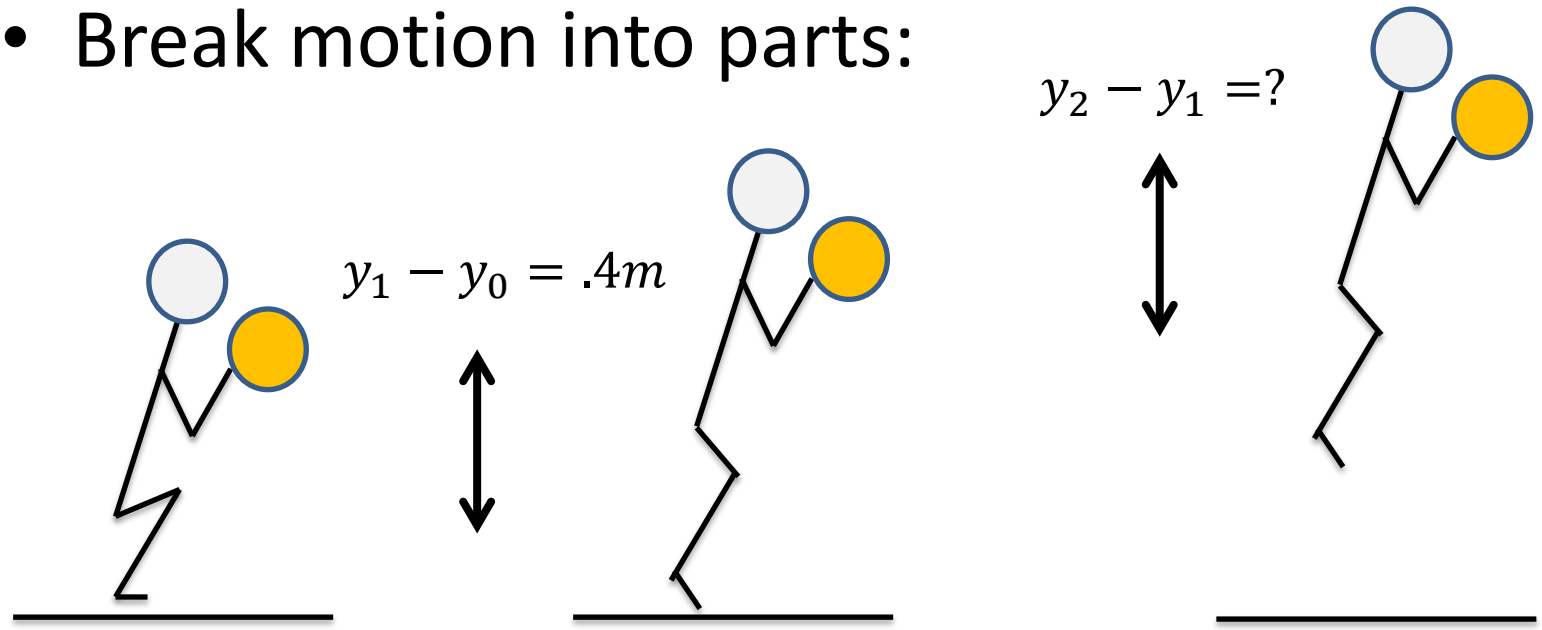
Use + root, since initial velocity must be upward

Example 4

- A basketball player is jumping to make a shot.
- His acceleration changes in the middle of his motion. We break his motion into two parts:
  - While in contact with the floor, the floor exerts a force against him: his acceleration is constant, but does not equal –g.
  - After leaving floor, gravity is now only force. So now a = -g
- Assume: While in contact with floor $a = +20\,\frac{m}{s^2}$. He accelerates through distance $y - y_0 = .4\,m$ while in contact with the floor.
- How high does he go after leaving the floor?

Example 4 (continued)

- Break motion into parts:



At lowest point of crouch

t_0

$y_0 = 0$

$v_0 = 0$

At minimum height

Just leaving ground

t_1

$y_1 = .4\ m$

$v_1 = ?$

At maximum height

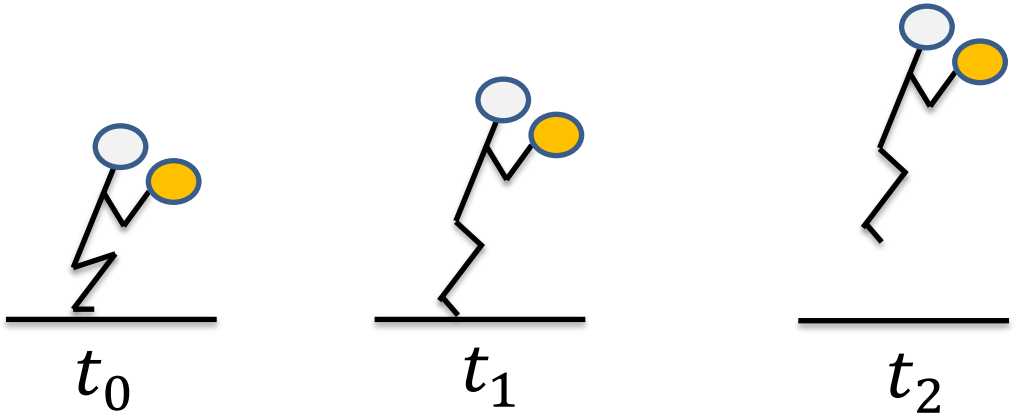
t_2

$y_2 = ?$

$v_2 = 0$

At maximum height

Example 4 (continued)



- From t_0 to t_1:

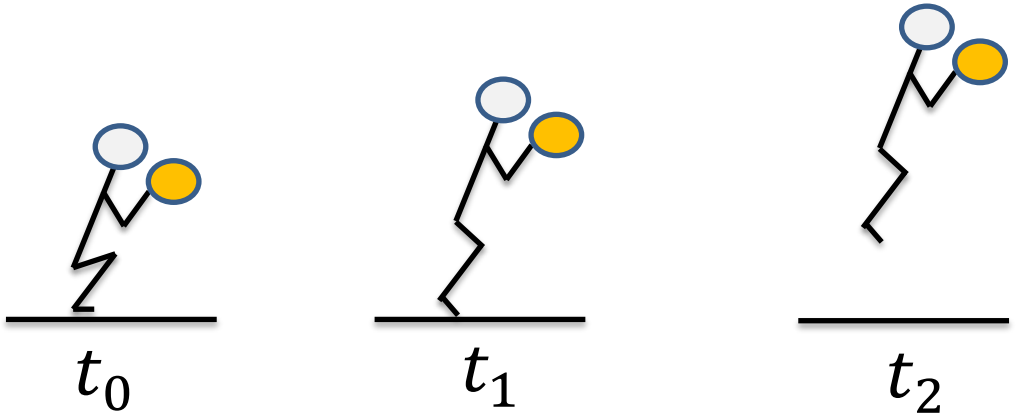
$$v^2 = v_0^2 + 2a(y - y_0)$$

$$v_1 = +\sqrt{v_0^2 + 2a(y_1 - y_0)} = +\sqrt{0 + 2(20\ \frac{m}{s^2})(.4\ m)} = \boxed{4.0\ \frac{m}{s}}$$

Upward

This is the velocity as he leaves the ground. Now look at the second part of the motion, from t_1 to t_2

Example 4 (continued)



- From t_1 to t_2:

At maximum height

Start same was as before, but now:

$$a = -g\,, \quad \text{initial speed is } v_1 = 4.0\ \frac{m}{s}, \quad \text{and} \quad v_1 = 0$$

So: $v_2^2 = v_1^2 - 2g(y_2 - y_1)$

$$y_2 - y_1 = \frac{v_2^2 - v_1^2}{-2g} = \frac{0 - \left(4.0\ \frac{m}{s}\right)^2}{-2\left(9.8\frac{m}{s^2}\right)} = \boxed{.82\ m}$$

(about 2.68 ft)